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New models of explosively accelerated projected shells☆

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Abstract

The asymptotic basis of models proposed earlier for the effective description of the acceleration of a soft metal shell which is accelerated by an explosion is given. A model of an incompressible liquid-crystal layer possessing linear longitudinal elasticity is considered as the three-dimensional medium. The equations of the shell are derived under assumptions concerning the smoothness or irregularity of the loaded surface of a layer. In the first case, a model of the inertial acceleration of a shell is obtained in the basic approximation and, in the second case, a model of a weakly elastic shell which refines it. The derivations of the asymptotic approach are specifically traced taking the example of a spherical layer. A dispersion relation is presented in the case of the planar problem, which indicates the existence of a finite range of wavelengths of increasing amplitude which can be used to create favourable wave conditions for the development or suppression of instabilities. A solution of problem of three-wave resonance is given.

1. Introduction

Applications of a model of an ideal incompressible fluid to the problem of the explosive projection of thin metal shells with the onset of a cumulative effect are widely known.[1–3] A review of the corresponding papers is available.[4] However, in specific, relatively complex situations, it is desirable to have simpler models (the inertial model[5,6] is such a model, for example). At the same time, experiments show that, in the development of instability of the form of a projected shell (which is mainly associated with Rayleigh–Taylor instability), other effects clearly manifest themselves, which cannot be described either by an inertial model or by a model of an incompressible fluid.

Experiments to investigate the motion of an initially almost planar layer of an easily deformable material which is accelerated by a large pressure overfall (in particular, of a circular copper plate with a pattern of the type of a triangular mesh of corresponding size applied to it) show that, under these conditions, there is a stably reproducible growth in the perturbations of a specific wavelength of the order of several thicknesses of the layer.[6]

In the series of experiments which were carried out, the characteristic magnitude of the pressure overfall was significantly greater than the elastic limit of the material but much smaller than the value of Young's modulus. Under these conditions, the model of a "weakly elastic" shell was proposed which assumes that the process of adaptability of the elastoplastic material occurs at the initial stage of acceleration, as a result of which the effective longitudinal elasticity of a shell with an elastic modulus proportional to the pressure difference on its two sides is depleted.

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The action of this elasticity is analogous to the action of surface tension. It only hinders the expansion of the shell, which is similar to the properties of a three-dimensional Treloar medium, a model of which has been derived from statistical considerations concerning the structure of rubber.[8] The physical nature of the elastoplastic processes which occur is not considered in the model and, as a consequence of this, it is to be considered as a semi-empirical model based on a hypothesis of a phenomenological nature.

When the shell is deformed, its self-intersections are possible. From the inset of the onset of self-intersection of the shell, an inelastic impact model[6] is used to describe the motion of its colliding parts. The study of certain simple exact solutions shows that, during expansion over long times, the shell reaches a stage of purely inertial dispersion.

An inertial force acts in the frame of reference associated with the middle element of the layer being accelerated, and this force, roughly speaking, does not give rise to a loss in stability of the front side of the shell but it makes its rear surface (which is turned towards the accelerating gas) unstable, from which significant parts of the material can take off. In this case, two forms of cumulation are possible: the cumulation of momentum and energy by the bars or plates ("fingers"), which are lagging behind as a result of the adhesion of the leading front as well as a cumulation of the specific (per unit mass) energy accompanying the decreasing thickness and momentum of the expanding convex formations ("bubbles").

In this model, as in the model of inertial motion,[6,9] the dynamic planar problem reduces to a system of linear equations and can be effectively investigated. In particular, in the case of a homogeneous shell, the qualitative behaviour of the solutions is determined by the corresponding dispersion equation, which shows the following. The behaviour of the short waves is of a stable oscillatory nature (naturally, for the theory to be applicable the wavelength must be much greater than the characteristic thickness of the shell, which cuts off the spectrum of wavelengths from below). As the wavelength increases, it reaches a critical value at which the frequency of the oscillations vanishes. At greater wavelengths, the perturbations increase without limit. This range of wavelengths contains a wavelength, equal to twice the critical wavelength with the greatest growth increment, the "resonance" wavelength. With a further unlimited increase in the wavelength, the increment falls to zero.

In the spatial case, the system of equations is quadratically non-linear due to the action of the external load. The effect, observed in the first experiments, of the appearance of six depressions, which are correctly arranged on the obstacle against which the deformed shell impacts, corresponds to a three-wave resonance, which is characteristic in the case of a quadratic non-linearity. A reduction in the thickness of the circular plate by a factor of two (and, correspondingly, the step size of the mesh of the rulings drawn on the plate) with the same remaining parameters led to the appearance of a second series of 18 depressions of a correspondingly smaller depth and with half the distance between their centres.

Preliminary drawing of the depressions on the surface of the plate being accelerated with a scale corresponding to the critical wavelength or somewhat smaller wavelengths promoted stabler behaviour of the plate during its acceleration. As a consequence of this, a broad depression with an almost flat bottom is formed on the obstacle after the impact of the plate. In the case of a thickness of the obstacle comparable with the diameter of the plate, splitting off from its opposite side is observed, which is also indicative of a planar impact.

It might be thought that the understanding which has been gained of the dynamic processes accompanying the motion of easily deformed shells enables one to increase the reproducibility and the effectiveness of the different practical devices for producing cumulative jets and other strikers, which are often unstable.[4]

The mathematical side of the inertial model has been thoroughly investigated earlier[7,10] with the solution of a large number of problems. A general review is available together with experimental data on overcoming the instability of the accelerating.[11]

The basic result of this paper is the asymptotic substantiation of models of an inertial and weakly elastic shell of infinitesimal thickness (models of a material surface) by analysing the dynamic behaviour of a thin three-dimensional layer of an incompressible liquid-crystal material as its thickness tends to zero. It is shown that, in the case of a sufficiently smooth side of a shell which is subjected to a load, the inertial model is realized in the basic asymptotic approximation. This, in particular, also refers to a layer of an ideal incompressible fluid and, for such a medium, the following approximations can be obtained in quadratures after the solution of the inertial problem has been obtained. If, however, the surface of the loaded side of the shell is uneven, which can be useful, for example, in the creation of favourable wave conditions for deformation, the model of a weakly elastic shell is the basic approximation.

2. Equations of motion of the shell

We consider a model of the motion of a thin shell which is subjected to a specified unilateral pressure p created, for example, by the detonation wave which is formed after the blasting of a layer of explosive adjacent to the shell. Some estimates of the quantities that are attainable here can be obtained from the solution of the one-dimensional Hugoniot problem of gas dynamics concerning the motion of a homogeneous planar piston ejected by a homogeneous gas which is initially at rest. This solution shows that the pressure p and the piston velocity υ vary according to the formulae

$$p = p_0\left(1 - \frac{(\gamma-1)\upsilon}{2a_0}\right)^{\frac{2\gamma}{\gamma-1}} = p_0\left(1 + \frac{(\gamma+1)\tau}{2}\right)^{-\frac{2\gamma}{\gamma+1}}, \quad \tau = \frac{tp_0}{\sigma a_0} \tag{2.1}$$

where a is the velocity of sound in the accelerating gas, γ is its adiabatic exponent, σ is the surface density of the piston and t is the time; initial quantities are labelled with a zero subscript. Hence, the gas pressure can be assumed to be constant and equal to the initial pressure p_0 so long as $\tau \ll 1$. In the case of an exponent γ which is sufficiently close to unity and when this inequality is satisfied, the piston velocity can be of the order of the initial velocity of sound. The acceleration of the piston is approximately equal to $g_0 = p_0/\sigma$ and the law of motion of the piston has the form $x \approx g_0 t^2/2$.

We introduce the Cartesian system of coordinates x^i ($i = 1, 2, 3$) and suppose $\mathbf{r}$ is the radius vector of the particles of the shell with the Euler components x^i. The equations of motion of a weakly elastic shell will then have the form[6,9]

$$\sigma\mathbf{r}_{tt} = p_0\mathbf{n} + \nabla_\alpha(\sigma c^2 a_0^{\alpha\beta}\mathbf{r}_\beta) \tag{2.2}$$

Here $\mathbf{n}$ is the vector of the unit normal directed towards where there is no pressure. A time derivative for constant Lagrange coordinates ξ^α ($\alpha = 1, 2$) is denoted by the subscript t and a derivative with respect to ξ^β is denoted by the subscript β. The symbol ∇ denotes a covariant derivative along a surface.

The pressure p_0 is henceforth assumed to be constant (it can also be assumed to be a function of t). The quantity c^2 is a characteristic of the longitudinal elasticity of the shell material, which has been mentioned in Section 1. It is introduced by the formula $c^2 = \kappa p_0/\rho$, where $\rho(\xi^\alpha)$ is the density of the material of the shell (which is assumed to be incompressible), κ is a dimensionless coefficient of the order of unity and the quantity c^2 can be provisionally called the square of the velocity of sound in the shell material. We also introduce the shell thickness $h = \sigma/\rho$.

Note that we obtain the model of purely inertial acceleration[6] by formally putting $c = 0$ in Eq. (2.2).

Suppose $a_{\alpha\beta}$ are the components of the metric tensor of the shell surface

$$a_{\alpha\beta} = \delta_{ij}x^i_\alpha x^j_\beta, \quad x^i_\alpha = \partial x^i/\partial\xi^\alpha \tag{2.3}$$

Here δ_{ij} is the Kronecker delta and x^i_α are the components of the tangential surface vectors $\mathbf{r}_\alpha$ such that $\mathbf{n} = \mathbf{r}_1 \times \mathbf{r}_2/|\mathbf{r}_1 \times \mathbf{r}_2|$, where

$$|\mathbf{r}_1 \times \mathbf{r}_2| = (\det(a_{\alpha\beta}))^{1/2} \equiv \sqrt{a}$$

The initial contravariant components of the surface metric tensor, which are used with the coefficient c^2 as the components of the specific elastic constant tensor, are denoted by the symbol $a_0^{\alpha\beta}$.

Note that all the elements of the theory of finite deformation of a shell (without linearization) are successively taken into account here; although, in the final analysis, the elastic component of the stress tensor in Eq. (2.2) is obtained as a linear component. In fact, by virtue of the law of conservation of mass, the Lagrange formula

$$\sigma = \sigma_0\sqrt{a_0}/\sqrt{a} \tag{2.4}$$

holds. As above, the corresponding initial quantities, that is, the functions ξ^α, are labelled with a zero subscript. Using formulae (2.2) and (2.4), we obtain the vector equation

$$\sigma_0\sqrt{a_0}\mathbf{r}_{tt} = p_0\mathbf{r}_1 \times \mathbf{r}_2 + (\sqrt{a_0}\sigma_0 c^2 a_0^{\alpha\beta}\mathbf{r}_\beta)_\alpha \tag{2.5}$$

Eq. (2.5) can be obtained from a holonomic variational equation with the Lagrangian

$$\Lambda = \sigma_0\sqrt{a_0}\frac{|\mathbf{r}|^2}{2} + \frac{p_0}{3}\mathbf{r}(\mathbf{r}_1\times\mathbf{r}_2) - \sigma_0\sqrt{a_0}\frac{c^2}{2}(a_0^{\alpha\beta}a_{\alpha\beta} - 2) \tag{2.6}$$

by varying the law of motion x^i (ξ^α, t). On the right-hand side of this equality, the second term with the opposite sign is the potential energy of the pressure forces, which depends on the derivatives of the law of motion of the shell. The last term corresponds to the positive elastic energy of the shell in which the strain tensor $(a_{\alpha\beta} - a^0_{\alpha\beta})/2$ appears in a linear form (we recall that $(a_0^{\alpha\beta}) = (a^0_{\alpha\beta})^{-1}$ like matrices). This formulation of the model can serve as a basis for developing approximate numerical methods for solving problems and for deriving conservation laws associated with divergent forms of equations and, also, for the correct specification of boundary conditions.

We also present the equation for the change in the energy of the shell or, more accurately, the equation for the kinetic energy, which does not take account of thermodynamic effects. On convoluting Eq. (2.5) with $\mathbf{r}_t$, we obtain

$$\begin{aligned}&\left(\sigma_0\sqrt{a_0}\frac{|\mathbf{r}_t|^2}{2} - \frac{p_0}{3}\mathbf{r}(\mathbf{r}_1\times\mathbf{r}_2) + \sigma_0\sqrt{a_0}\frac{c^2}{2}(a_0^{\alpha\beta}a_{\alpha\beta} - 2)\right)_t - \\ &-\frac{p_0}{3}[(\mathbf{r}(\mathbf{r}_t\times\mathbf{r}_1))_2 - (\mathbf{r}(\mathbf{r}_t\times\mathbf{r}_2))_1] - (\sqrt{a_0}\sigma_0 c^2 a_0^{\alpha\beta}\mathbf{r}_\beta\cdot\mathbf{r}_t)_\alpha = 0\end{aligned} \tag{2.7}$$

In the case of a shell with an edge, we shall assume that the shell is fitted into a channel of the corresponding shape, that is, it is a deformable piston, in order to avoid having to consider the possible inflow of gas onto the front surface of the shell and an analysis of the boundary rarefaction waves which arise in this case.

When the trajectories of the particles of the shell intersect, the model of an absolutely inelastic impact is used, which corresponds to the summation of the momentum vectors of the colliding point masses. In this case, depending on the collision geometry, both new surfaces as well as bars or even separate point masses can be formed. Here, however, it is necessary to take account of the fact that, in the case of a subsonic collision process (with respect to the velocity of sound propagation through the shell), it will be accompanied by the emission of perturbations of the form of the shell which depart in front of a weak or a strong kink with a smooth or only continuous transition respectively. Discontinuities in the shell thickness h, which are expressed in terms of a, can also propagate.

By virtue of the linear form of the terms of the hyperbolic system of Eq. (2.5), which contain higher (second) derivatives of the radius vector $\mathbf{r}$, weak discontinuities will propagate relative to the particles of the shell at the velocity of sound, the magnitude of which is associated with the quantity c. Without the risk of not arriving at a solution, the same can also be assumed in this case with respect to strong discontinuities, which correspond to a jump in the first derivatives when there are no concentrated actions. Of course, everything depends on the formulations of the specific problems.

More accurately, the characteristic function $f(\xi^\alpha, t)$, determining the equation of the characteristics $f = 0$ along which discontinuities are possible, satisfies the equation

$$f_t^2 = c^2 a_0^{\alpha\beta} f_\alpha f_\beta \tag{2.8}$$

The solution of Eq. (2.8) is associated with the determination of the corresponding bicharacteristics (rays), the equations of which contain the relative velocity of propagation of perturbations

$$\frac{d\xi^\alpha}{dt} = \frac{f_t a_0^{\alpha\beta} f_\beta}{a_0^{\mu\nu} f_\mu f_\nu} = \pm\frac{c a_0^{\alpha\beta} f_\beta}{(a_0^{\mu\nu} f_\mu f_\nu)^{1/2}} \tag{2.9}$$

In Euler variables, the velocity of a discontinuity $dx^i/dt = x^i_t + x^i_\alpha d\xi^\alpha/dt$. Hence, the magnitude of the spreading of the relative velocity of the discontinuity (2.9), measured in the actual metric $a_{\alpha\beta}$, can differ considerably from the magnitude of c in the case of large deformations of the shell.

We also note that the possible integral forms of the kinetic energy Eq. (2.7) cannot, generally speaking, serve to derive the energy conditions in a discontinuity. They can, in the best case, only give the magnitude of the dissipation of the sum of the kinetic and elastic energies in a discontinuity. We omit the analysis of the total energy equation and possible formulations of thermodynamic problems.

3. A three-dimensional model of an accelerate layer

We will proceed to substantiate the equations of a weakly elastic shell (2.5) by means of an asymptotic analysis of the motion of a three-dimensional, incompressible, homogeneous, liquid-crystal layer.

Suppose the layer is under the action of a unilateral, spatially homogeneous pressure $p_0(t)$. It is convenient to consider the layer surface S_0, on which there is no pressure, as a reference surface from which we shall measure the layer thickness h. In the accelerating reference system which is locally associated, for example, with the middle surface of the layer, the surface S_0 is stable and the surface S_1 is unstable in a Rayleigh–Taylor sense.[4] Hence, the use of the surface S_0 as a reference surface significantly simplifies the formulae and makes the asymptotic analysis more precise. Moreover, when wave effects are used for the purposeful development of instability or for its suppression in the experiments which have been carried out and in applications,[11] the surface S_0 was made smoother.

At the initial instant of time within the layer in the neighbourhood of the surface S_0, we introduce the Lagrange coordinates ξ^α and ζ, $\zeta \in (0, 1)$ such that

$$x_0^i = r_0^i(\xi^\alpha) + n_0^i h_0(\xi^\alpha)\zeta \qquad (3.1)$$

where x_0^i are the initial Cartesian coordinates. The coordinate ζ is related to the usual normal coordinate ξ by the formula $\xi = \zeta h_0$. Here, for convenience, the vector of the normal $\mathbf{n}_0$ is directed towards the increase in pressure. On the surface S_0, we have $\zeta = 0$ and, on the surface S_1, to which the pressure is applied, $\zeta = 1$. Note that, in the Lagrange coordinates ξ^α, ζ, the vector of the normal $\mathbf{n}$ always has the covariant components (0, 0, 1) on all of the surfaces $\zeta = \text{const.}$ apart from unimportant normalization.

Calculating the Jacobian for the transformation from the initial Cartesian coordinates x_0^i to the variables ξ^1, ξ^2, ζ, we obtain

$$\Delta_0 \equiv \in_{ijk} x_{01}^i x_{02}^j x_{0\zeta}^k = \sqrt{a_0} h_0 (1 - \zeta h_0 b_{0\alpha}^\alpha + \zeta^2 h_0^2 \det(b_{0\beta}^\alpha)) \qquad (3.2)$$

where $b_{0\beta}^\alpha = -r_{0\beta}^i \nabla_0^\alpha n_{0i}$ is the initial tensor of the external curvature of the surface S_0 and $\in_{ijk}$ is a completely antisymmetric Levi-Civita symbol.

The ratio of the layer thickness to the characteristic longitudinal linear dimension, for example, to the initial radius of curvature of the surface S_0, is the small parameter in the problem. In particular, we can put

$$\varepsilon = \max(h_0|\mathbf{b}_0|) \ll 1 \qquad (3.3)$$

The modulus of the external curvature tensor is denoted by $|\mathbf{b}_0|$: $|\mathbf{b}_0| = (b_0^{\alpha\beta} b_{0\alpha\beta})^{1/2}$. Usually, x_0^i which is not very small.

The equations of motion of the particles of the layer, under the assumption that the material is incompressible, which determine the law of motion of the medium $x^i = x^i(\xi^\alpha, \zeta, t)$ with a constant density ρ, have the form

$$x_{i,tt} x_\alpha^i + \frac{p_\alpha}{\rho} = \frac{1}{\Delta_0}(\Delta_0 C^{\beta\gamma} x_{i,\beta})_\gamma x_\alpha^i,\ x_{i,tt} x_\zeta^i + \frac{p_\zeta}{\rho} = \frac{1}{\Delta_0}(\Delta_0 C^{\beta\gamma} x_{i,\beta})_\gamma x_\zeta^i;\ \Delta_0 = \in_{ijk} x_1^i x_2^j x_\zeta^k \qquad (3.4)$$

Suppose the components of the tensor $\mathbf{C}$, which characterizes the longitudinal elasticity of the layer, in these Lagrange coordinates have the form

$$C^{\alpha\beta} = (\kappa p_0/\rho) a_0^{\alpha\beta} \qquad (3.5)$$

that is, it is proportional at each point of the layer to the initial metric tensor of the reference surface S_0. The pattern of the stratification of a metal layer after a sequence of shock waves and rarefaction waves have passed through it[4] can serve as a physical basis for the introduction of this model.

Hence, according to the classification of the continuous symmetry groups of liquid crystals,[12] we are dealing with a medium which possesses a local symmetry group (a group for which the stress tensor is insensitive to affine deformations) $G_{3.11}(x=0)$. The matrices of the corresponding transformation when $a_0^{\alpha b} = \delta^{\alpha\beta}$ have the form

$$\begin{Vmatrix} \cos\varphi & -\sin\varphi & l \\ \sin\varphi & \cos\varphi & m \\ 0 & 0 & 1 \end{Vmatrix}, \quad \varphi \in [0, 2\pi] \tag{3.6}$$

where l and m are arbitrary numbers. So, the insensitivity group consists of rotations in the plane of the variables ξ^1 and ξ^2 and all possible simple displacements along this plane. Here, the vector of the normal to the surface $\zeta = \text{const}$ passes into itself (its covariant Lagrange components remain invariant).

Media of this kind are treated as liquid crystals of the smectic type possessing a lamellar structure. A pack of smooth playing cards serves as an obvious example of such a medium. The specific internal elastic energy is selected by the linear function of the strain tensor (the quadratic function of the distortion $x_p^i = \partial x^i/\partial\xi^p$, $p = 1, 2, 3$)

$$U = C^{pq}(g_{pq} - \overset{0}{g}_{pq})/2$$

where $g_{pq} = \delta_{ij}x_p^i x_q^j$ are the Lagrange covariant components of the metric tensor.

The medium which has been introduced above is similar to an isotropic Treloar medium that models the properties of rubber.[8] In the Treloar model of a medium $C^{pq} = Kg_0^{pq}$ and K is a constant of the material, but, in the case being considered here, the tensor $\mathbf{C}$ is assumed to be degenerate (a covector $\mathbf{N}$ exists such that $C^{pq}N_q = 0$, $\mathbf{N} = (0, 0, 1)$), which is adapted to describe just the longitudinal elastic properties of a shell. The given symmetry group, in addition to the invariant $C^{pq}g_{pq}$ and the invariant $g = \det(g_{pq})$, associated with the incompressibility condition $g = g_0 = \Delta_0^2$, has a further invariant $g^{pq}N_pN_q = g^{\zeta\zeta}$. However, introduction of the latter invariant into the internal energy leads to an increase in the non-linearity of the equations. When account is taken of incompressibility, it depends on four degrees of distortion which leads, at least, to cubic non-linearity of the equations of motion.

The physical meaning of these invariants, if they occur in the internal energy with positive coefficients is as follows: the invariant $C^{pq}g_{pq}$ describes the resistance of the material to longitudinal stretching and $g^{\zeta\zeta}$ describes the resistance of the material to transverse compression. The incompressibility condition transfers the reactions of the material into its opposite properties in a perpendicular direction. The layer material being described therefore only responds to the isotropic part of the longitudinal extension and to transverse compression.

The introduction of an external pressure p_0 into the tensor $\mathbf{C}$ (3.5) is related to the hypothesis of the appropriate adaptability of the material as a result of plastic deformation,[13] but can be formally balanced by the addition of a quantity κ, which can be attributed to the properties of the material. It is surprising that experiments on the explosively accelerated copper plates show that the quantity $\kappa = 1.03$,[7] that is, it is practically equal to zero.

When the layer has no elasticity $C^{pq} = 0$, we have the usual equation for an incompressible fluid. Moreover, in formulating the boundary conditions when $C^{pq} \neq 0$ on the surfaces S_0 and S_1 of the form $[p_i^j n_j] = 0$, the elastic part of the stress tensor $p_{(e)}^{mm} = 2\rho\partial U/\partial g_{mn}$ (except for the usual terms with pressure in the case of an incompressible fluid) with the components

$$p_{i(e)}^{\alpha} = C^{\beta\alpha}x_{i,\beta}, \quad p_{i(e)}^{\zeta} = 0 \tag{3.7}$$

automatically drop out. Hence, there are only boundary conditions for the pressure: $p = 0$ when $\zeta = 0$ and $p = p_0$ when $\zeta = 1$.

Note that the introduction of the non-quadratic invariant $g^{\zeta\zeta}$ into the theory also makes no contribution to the shear surface force on the layer boundaries but leads to the absence of pressure continuity on its surfaces.

4. Asymptotic analysis of the equations of the layer

Starting out from the form of equations (3.2) and (3.4) and taking account of the fact that the small initial thickness of the layer h_0 of the order of ε occurs in a polynomial form in the condition for the incompressibility of the material, we shall seek a solution in the form

$$x^i = r^i(\xi^\alpha, t, \varepsilon) + \varepsilon y^i(\xi^\alpha, \zeta, t, \varepsilon), \quad p = p(\xi^\alpha, \zeta, t, \varepsilon) \tag{4.1}$$

where r^i, y^i and p are functions of the order of unity. Generally speaking, an additional dependence of these functions on ε can arise in satisfying the global condition $p = p_0$ when $\zeta = 1$ in a specific approximation. By virtue of the definition of the surface S_0, the functions y^i are equal to zero when $\zeta = 0$.

Substituting these expressions into Eq. (3.4) and discarding necessarily small terms containing ε^2 we find that it is necessary to formulate additional assumptions. From the second equation of (3.4) we have

$$r_{i,tt}y^i_\zeta + \frac{p_\zeta}{\varepsilon\rho} = \frac{1}{h_0\sqrt{a_0}}(h_0\sqrt{a_0}C^{\beta\gamma}r_{i,\beta})_\gamma y^i_\zeta \tag{4.2}$$

Since the key function for the pressure is $p = p_0\zeta$, the second derivatives r_i with respect to t and with respect to ξ^γ must, generally speaking, be of the order of $1/\varepsilon$. Two cases are possible here. We shall confine ourselves to a Cauchy problem with zero initial velocity. Then, if, at the initial instant of time, the surfaces S_0 and S_1 are sufficiently smooth, then it can be assumed that all the functions being considered depend on the ratio $\tau = t/\sqrt{\varepsilon}$ and that the derivatives with respect to τ and the remaining variables are of the order of unity. Hence, the stretched out time τ is the natural time for this system. In the case of a not very smooth initial form of the surface S_1, which is associated with the corresponding form of the function h_0 (the surface S_0 is always assumed to be smooth), me must also propose a dependence (with derivatives of the order of unity) on the stretched surface coordinates $\eta^\gamma = \xi^\gamma/\sqrt{\varepsilon}$.

We will now consider the first case when the layer surface is sufficiently smooth. The validity of the assumptions which have been introduced will be demonstrated in the following section on the basis of the exact solution of a problem on the acceleration of a spherical layer.

We define the thickness of the thin layer as $h = h_0\sqrt{a_0}\sqrt{a}$. Using the assumptions made, from the first equation of system (3.4) we obtain

$$r_{i,\tau\tau}r^i_\alpha = 0$$

From the second equation of (3.4), on integrating with respect to ζ, we have

$$n_iy^i = \frac{h}{\varepsilon}\zeta \tag{4.3}$$

After integrating with respect to ζ, Eq. (4.2) also takes the form

$$r_{i,\tau\tau}y^i + p = 0$$

Eliminating the functions y^i from the last equation, we obtain

$$\rho r_{i,\tau\tau}n^ih\zeta + \varepsilon p = 0$$

Actually, from this relation we have

$$p = p_0\zeta \tag{4.4}$$

If the surface density is defined as $\sigma = \rho h$, then, as a result, we obtain the so-called inertial model of an accelerated shell[6]

$$\sigma\mathbf{r}_{tt} = -p_0\mathbf{n}, \quad \sigma = \sigma_0\sqrt{a_0}/\sqrt{a} \tag{4.5}$$

Moreover, in the following approximation, from the first equation of system (3.4) we obtain an equation determining the remaining functions y^i

$$r_{i,\tau\tau}y^i_\alpha + y_{i,\tau\tau}r^i_\alpha = \frac{1}{h_0\sqrt{a_0}}(h_0\sqrt{a_0}C^{\gamma\beta}r_{i,\gamma})_\beta r^i_\alpha$$

Using the expansion

$$y^i = n^i\zeta\frac{h}{\varepsilon} + r^i_\alpha y^\alpha \tag{4.6}$$

after some reduction we obtain

$$(a_{\alpha\beta}y^\beta_\tau)_\tau = -\frac{\zeta}{\varepsilon}(n_ih)_{\tau\tau}r^i_\alpha + \frac{1}{h_0\sqrt{a_0}}(h_0\sqrt{a_0}C^{\gamma\beta}r_{i,\gamma})_\beta r^i_\alpha \tag{4.7}$$

Eq. (4.7) shows that, after solving the basic system (4.5) the remaining terms can be successively calculated in quadratures. All the successive approximations can be determined in a similar manner. Note that, in the absence of elasticity, the problem of the accelerating of a layer of an incompressible fluid is also thereby solved.

We will now consider the second case when the initial form of the surface S_1 is not completely smooth. In this case, using the extended surface variables $\eta^\alpha = \xi^\alpha/\sqrt{\varepsilon}$, we obtain that formulae (4.3) and (4.4) for $n_i y^i$ and p are practically preserved but Eq. (4.5), when account is taken of the definition of the surface density σ, takes the form

$$\sigma\mathbf{r}_{tt} = -p_0\mathbf{n} + \nabla_\alpha(\sigma C^{\alpha\beta}\mathbf{r}_\beta) \tag{4.8}$$

Consequently, in this case we arrive at the model of a weakly elastic shell considered in Section 2.

For the quantities y^α, we correspondingly obtain a hyperbolic system of equations, which we will not write out here.

Hence, the need to take account of the residual elasticity of a shell already in the first approximation is associated with the possibility of effectively taking account of the initial very small-scale perturbations on the side of the layer which is subjected to the action of an external pressure. It will be shown below that the stability on instability of the motion of the layer as a whole, in accordance with the dispersion and non-linearity properties of Eq. (4.8), depends very much on the initial pattern of the perturbations.

5. Solutions with spherical symmetry

In order to obtain an idea of the validity of the asymptotic method, we will consider the spherically-symmetric problem of the pressing of a layer of an incompressible lamellar material within the framework of system (3.4) considered in Section 3. We will confine ourselves to the problem of the acceleration of a homogeneous layer of mass M, the points of which have zero initial velocity, under the action of a constant external pressure p_0, neglecting the pressure drop of the propellant gas.

We will change from the Cartesian coordinates x, y and z to the spherical coordinates r, θ and φ:

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta$$

Suppose $r = R_0(t)$ and $r = R_1(t)$ are the boundaries of the layers S_0 and S_1 respectively.

By virtue of the spherical symmetry, the variables θ and φ are Lagrange variables. We also introduce the mass variable m, which is associated with the variable ζ (3.1) by the formula

$$\zeta = \frac{1}{h_0}\left(\frac{3m}{4\pi\rho}\right)^{1/3} = \frac{r_0 - R_0(0)}{h_0}$$

where r_0 is the initial radius of the particles of the layer.

In spherical coordinates, the tensor $\mathbf{C}$ has the following non-zero components:

$$C^{\theta\theta} = \frac{K}{R_0^2(0)}, \quad C^{\varphi\varphi} = \frac{K}{R_0^2(0)\sin\theta}, \quad K = \frac{\kappa p_0}{\rho} = c^2$$

The Jacobian $\Delta_0 = r_0^2\sin\theta$.

The incompressibility condition, from which the conservation of volume follows, determines the form of the function $r = r(m, t)$:

$$r = \left(R_0^3(t) + \frac{3m}{4\pi\rho}\right)^{1/3} \tag{5.1}$$

The equation in the radial direction (3.4) has the form

$$r_{tt} + 4\pi r^2 p_m + \frac{2Kr}{R_0^2(0)} = 0 \tag{5.2}$$

On substituting $r(m, t)$ and integrating with respect to m, after eliminating of m we obtain the pressure distribution

$$\frac{p}{\rho} = (R_0^2\dot{R}_0)^\cdot\left(\frac{1}{r} - \frac{1}{R_0}\right) - \frac{R_0^4\dot{R}_0^2}{2}\left(\frac{1}{r^4} - \frac{1}{R_0^4}\right) - \frac{K}{R_0^2(0)}(r^2 - R_0^2) \tag{5.3}$$

The equation for the unknown function $R_0(t)$ follows from the condition $p=p_0$ when $r = R_1 = (R_0^3 + 3M/(4\pi\rho))^{1/3}$. The first integral of this equation can be obtained if the equation for the change in the total energy of the layer as a result of the work done by the pressure p_0 is considered. We therefore obtain

$$2\pi R_0^4\dot{R}_0^2\left(\frac{1}{R_0}-\frac{1}{R_1}\right)+\frac{4\pi K}{5R_0^2(0)}(R_1^5-R_1^5(0)-R_0^5+R_0^5(0)) = \frac{4\pi p_0}{3\rho}(R_0^3(0)-R_0^3) \tag{5.4}$$

Eq. (5.4) enables us to carry out the investigation of the limit for a small ratio $h_0/R_0(0)\sim\varepsilon$ easily.

The work of the external pressure p_0, on the right, is of the order of unity. The elastic energy of the layer (the second term on the left-hand side of the equation) here (in the case of a smooth surface S_1) is of the order of ε, actually small and can be neglected. Hence, by virtue of the relation $R_1 - R_0 \approx h_0$, the structure of the kinetic energy (the first term), which is of the order of unity, shows that the characteristic time of the motion of the layer is, in fact, of the order of $\sqrt{\varepsilon}$.

In the thin shell approximation $R=R_0\approx R_1$ within the framework of the inertial model in the case of compression, we have the following energy integral

$$\frac{\dot{R}^2}{2}+\frac{p_0}{3\sigma_0 R^2(0)}(R^3-R^3(0)) = 0; \quad \sigma_0 = h_0\rho\sim\varepsilon \tag{5.5}$$

which can be used to represent the solution in the form of a quadrature $t(R)$ and, when $R=0$, also enables us to determine the finite kinetic energy of the shell $M\dot{R}^2/2$, which is converted into heat during inelastic impact against the centre of symmetry, and the collapse time, which is defined by the relation

$$t_* = \left(\frac{3\sigma_0 R(0)}{2p_0}\right)^{1/2}\int_0^1\frac{d\rho}{\sqrt{1-\rho^3}} \tag{5.6}$$

It should also be kept in view that, by virtue of the incompressibility of the shell material, it is impossible to approach to the very centre $r=0$ and, in practice, the value of r must be confined to small radii

$$r\sim h\sim(h_0R_0^2(0))^{1/3}\sim\varepsilon^{1/3}$$

We also present an estimate of the applicability of the condition of the constancy of p_0 up to instant when the shell collapses. By virtue of formulae (2.1) and (5.6), we have the relation $\rho/\rho_g \gg R_0/h_0$, where ρ_g is the density of the accelerating gas, which is completely feasible in the case of a sufficiently dense material and a shell which is not too thin.

We now consider the problem of the expansion of a spherical layer under the action of an internal pressure. In this case, in Eq. (5.4) it is necessary to change the sign in front of the pressure p_0 and to interchange the positions of the functions R_0 and R_1. As a result, we obtain

$$2\pi R_0^4\dot{R}_0^2\left(\frac{1}{R_1}-\frac{1}{R_0}\right)+\frac{4\pi K}{5R_1^2(0)}(R_0^5-R_0^5(0)-R_1^5+R_1^5(0)) = \frac{4\pi p_0}{3\rho}(R_0^3(0)-R_0^3) \tag{5.7}$$

where $R_0^3 = R_1^3 + 3M/(4\pi\rho)$.

At first glance, it appears that it is necessary to take account of the restrictive character of the elastic forces. However, if we take the limit of a thin shell, taking account of the small ratio $h_0R_{(0)}^2/R^3$, we obtain the equation

$$\frac{\dot{R}^2}{2}-\frac{p_0}{3\sigma_0 R^2(0)}(R^3-R^3(0))+\frac{K}{R^2(0)}(R^2-R^2(0)) = 0 \tag{5.8}$$

which is exactly identical to the energy integral for the motion of a spherical, weakly elastic shell (4.8). By virtue of the fact that the elastic constant itself is depleted under the action of the external pressure $K\sim p_0/\rho$ (3.5), it is clear that terms associated with the elasticity are, in general, small for all times of the motion. Hence, in this case, the equations of motion within the framework of the inertial model of a shell (4.5) are also always valid provided the pressure p_0 can be assumed to be sufficiently high for the application of the assumption concerning the adaptability of the material.

Note that, if the pressure can be assumed to be constant during the expansion of the shell, then accentuation conditions, that is the departure of the shell radius to infinity after a finite time (or the unbounded cumulation of energy), occurs.

The time for the complete expansion is equal to

$$t_\infty = \left(\frac{3\sigma_0 R(0)}{2p_0}\right)^{1/2}\int_1^\infty \frac{d\rho}{\sqrt{\rho^3-1}} \tag{5.9}$$

The asymptotic form of the law of motion when $R\to\infty$ therefore has the form

$$R \approx \frac{6\sigma_0 R^2(0)}{p_0}\frac{1}{(t-t_\infty)^2}$$

In order to obtain a more practicable result in this problem, a bounded shell, homogeneous throughout its volume, and an adiabatic pressure drop

$$\frac{p_0(t)}{p_0(0)} = \left(\frac{R(0)}{R(t)}\right)^{3\gamma}$$

can be considered, where $4\pi R^3/3$ is the volume occupied by the gas and $\gamma>1$ is the adiabatic exponent. In this case, the energy integral will have the form

$$\frac{\dot{R}^2}{2} - \frac{p_0(0)R(0)}{3(\gamma-1)\sigma_0}\left(1-\left(\frac{R(0)}{R}\right)^{3(\gamma-1)}\right) = 0 \tag{5.10}$$

Eq. (5.10) shows that the shell radius $R(t)$ increases linearly when $R\to\infty$ and the critical energy $E = M\dot{R}^2/2$ reaches the value

$$E_\infty = \frac{4\pi R^3(0)p_0(0)}{3(\gamma-1)}$$

whic is equal to the initial energy of the gas.

Hence, also when account is taken of the pressure drop behind the shell, this solution shows the possibility of the effective action on obstacles of elements of the shell which are low in mass but with a high specific energy.

Solutions with spherical symmetry and their asymptotic forms play a considerable role in the study of the internal resonances of shells (see Section 7 below). These solutions can also serve as a test for the approval of different approximate methods.

6. The planar problem

In the case of the planar problem, we have $x^3=\xi^2$ and all the remaining variables are functions of ξ^1 and t. We will introduce the complex Euler variable $z=x^1+ix^2$ and, to simplify the equations, we will also use the following mass variable

$$\mu = \int \frac{\sigma_0 a_0^{1/2}}{p_0} d\xi^1$$

as a Lagrange coordinate. In this case, $\sigma=p_0/|z_\mu|$. The dimension of the variable μ is equal to the dimension of t^2.

The equations of motion then take the form

$$z_{tt} = iz_\mu + (\tilde{c}^2(\mu)z_\mu)_\mu, \quad \tilde{c}^2 = \kappa\rho h_0^2/p_0 \tag{6.1}$$

Note that the magnitude of the initial thickness of the shell h_0 can also be a variable.

The complex Eq. (6.1) is a system of two hyperbolic equations. However, even in the case of a constant magnitude of $\tilde{c}$, it possesses considerable dispersion which in many respects characterizes the unusual behaviour of the solutions for different initial data.

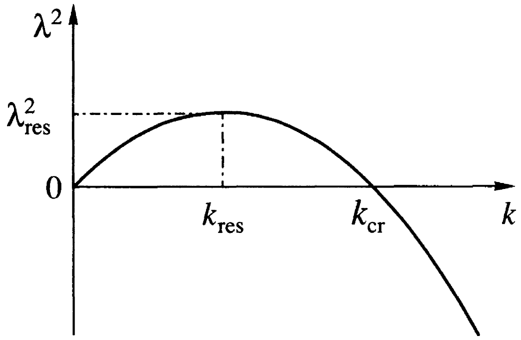

Fig. 1.

Actually, we shall consider an elementary solution in the case when $\tilde{c}$ = const, which corresponds locally to the deformation of a cylinder. Suppose

$$z = A_0\exp(\lambda t - ik\mu), \quad \sigma = \frac{p_0}{|A_0|k}\exp(-\mathrm{Re}\lambda t)$$

where A_0 and λ are complex constants and $k > 0$ is a real constant. We then have the following dispersion equation

$$\lambda^2 = k - \tilde{c}^2k^2 \tag{6.2}$$

which shows (Fig. 1) that there is a critical wavenumber $k_{cr} = 1/\tilde{c}^2$ corresponding to an equilibrium state $\lambda = 0$. In the case of wavenumbers $k < k_{cr}$, the constant λ is real, which corresponds either to an increase or an attenuation of the amplitude of the wave $A_0\exp\lambda t$ with time. There is a single maximum in the magnitude of λ^2 when $k = k_{cr}/2$. Hence, there is a wave with a mass length $4\pi\tilde{c}^2$, which is henceforth called "a resonance wave", with the most rapidly increasing amplitude, that characterizes the maximum instability of the process. In the conventional linear variables, this gives the relation

$$4\pi\kappa\sigma_0 h_0 = \int_0^{l_{res}} \sigma_0 a_0^{1/2} d\xi^1 = \mathrm{const} \tag{6.3}$$

which corresponds approximately (since the value of $\sigma_0 a_0^{1/2}$, generally speaking, is not constant) corresponds to a wavelength $l_{res} = 4\pi\kappa h_0$. When $k \to 0$, the rate of growth in the amplitude falls to zero. The quantity $\sigma_0 a_0^{1/2}$, for example, for a constant value of σ_0 and a small initial perturbation of the plane, differs from a constant by a quantity of the second order of smallness.

In the case when $k > k_{cr}$, the parameter λ becomes pure imaginary: $\lambda = i\omega$, where ω is the frequency of the amplitude oscillations, which is periodic in time.

In polar coordinates $z = r\exp(i\theta)$, the solution of the above type has the form

$$r = |A_0|\exp(\mathrm{Re}\lambda t), \quad \theta = \mathrm{Im}\lambda t - k\mu + \arg A_0$$

and, obviously, possesses cylindrical symmetry. When $\lambda^2 > 0$, it describes a monotonic expansion or compression and, when $\lambda^2 < 0$, the rotation of a homogeneous cylindrical shell.

By virtue of the linearity and homogeneity of Eq. (6.1), a solution, corresponding to rigid body motion of the type of Hugoniot solution (2.1), can also be added to it. It is then possible to speak about a more complex motion of a corrugated shell (a solution of this kind was been pointed out earlier in Ref. 6). In this case, close to the resonance wavelength, even initially small deviations of the form of the shell from planar will increase exponentially, outstripping the average motion with a constant acceleration.

As an example, we present the solution of the problem of a corrugated shell within the framework of the inertial model when it can be assumed that $\tilde{c} = 0$ or that k is small. In real variables ($z = x + iy$), we have

$$x = g_0\mu + A_0\mathrm{ch}\sqrt{k}t\sin k\mu, \quad y = g_0t^2/2 + A_0\mathrm{ch}\sqrt{k}t\cos k\mu$$

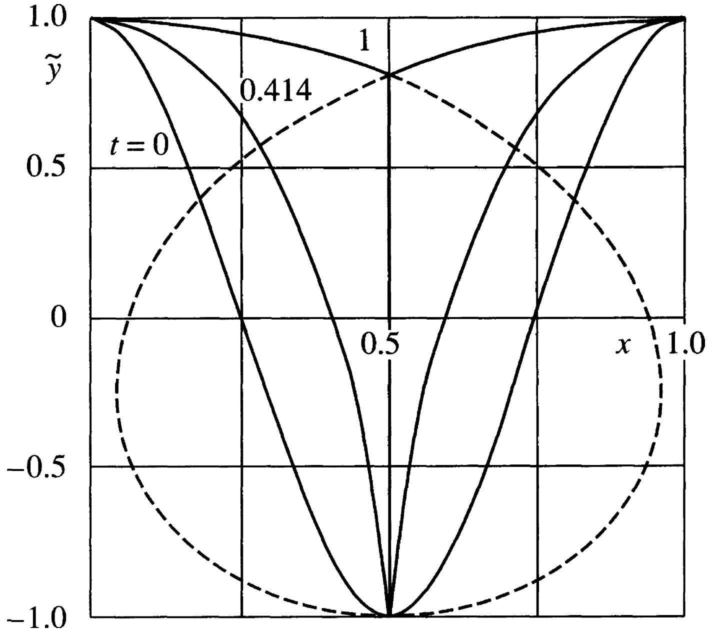

Fig. 2.

where g_0 and A_0 are real positive constants. If $A_0 \ll g_0/k$, the initial form of the shell can be assumed to be approximately cosinusoidal.

Calculation of the surface density gives

$$\sigma = p_0[(g_0 + A_0 k \operatorname{ch}(\sqrt{k}t)\cos k\mu)^2 + (A_0 k \operatorname{ch}(\sqrt{k}t)\sin k\mu)^2]^{-1/2}$$

With the passage of time, when $k\mu = \pi(1+2n)$, where n is an integer corresponding to $x = g_0\pi(1+2n)/k$, the density σ tends to infinity and a series of vertical "blades", lagging behind the main part of the shell, start to be formed. The action of the pressure p_0 on each blade is self-balanced. The particles of a blade then move inertially, acquiring a velocity which corresponds to the sum of the vertical components of the momentum of the particles of the initial shell (Fig. 2).

One period (with respect to the variable x) of the form of the shell $y = y(x, t)$ at different instants of time t is shown in Fig. 2. The dimensionless variables

$$\tilde{x} = \frac{xk}{2\pi g_0}, \quad \tilde{y} = \frac{y - g_0 t^2/2}{A_0 \operatorname{ch}(\sqrt{k}t)}$$

are used. In the calculations, the units of measurement are chosen such that $k = 2\pi$ and $g_0 = 1$. The parameter $A_0 = 0.1$. At the instant of time $t = 0.414$, a cusp is formed in which $\sigma = \infty$. The blade formed is denoted by a segment of the vertical line when $t = 1$. In this case, the "virtual" smooth continuation of the shell after its self-intersection (if the action of the pressure p_0 on the shell from within the closed domain were to continue here) is denoted by the dashed line.

7. Three-wave resonance

As the example of a spherically-symmetric solution (Section 5) shows, an even faster acceleration of a shell than an exponential acceleration, that is, an acceleration of the order of $1/(\Delta t)^2$, is possible. Can small perturbations of a plane lead to such an increase ? In particular, an experiment with accelerated circular plate, on the rear side of which diametral channels had been symmetrically made, indicated the formation of six depressions on the obstacle[6] which, as can be surmised, qualitatively corresponds to a wavelength with an extremally increasing amplitude. A three-wave internal resonance which is characteristic of a quadratic non-linearity[14] is at hand. We will show that it possesses the property of growth of the same order as the spherical solution.

We will assume that the initial perturbations are sufficiently small, such that the coefficients of Eq. (2.5) can be taken as being constant. Hence, we have

$$\mathbf{r}_{tt} = g_0(\mathbf{r}_1 \times \mathbf{r}_2) + c^2\delta^{\alpha\beta}\mathbf{r}_{\alpha\beta}; \quad g_0 = p_0/\sigma_0 \tag{7.1}$$

We now consider the problem of the acceleration of a plane in which a symmetric pattern of three standing waves is realized. We expand the perturbed part of the solution in triply periodic functions, corresponding to the triangular lattice, at the mesh points of which, for example, the maxima of a vertical (along $\mathbf{e}_3$) perturbation are located. Hence, there is a finite group of sixth-order rotations, which leaves a given point fixed, plus a group of corresponding translations.

The solution has the form

$$\begin{aligned} &\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_I + \mathbf{r}_{II}, \quad \mathbf{r}_0 = \mathbf{e}_\alpha\xi^\alpha + \mathbf{e}_3 g_0 t^2/2 \\ &\mathbf{r}_I = \operatorname{Re}\sum_{p=1}^{3}(A(t)\mathbf{e}_3 - iB(t)\mathbf{k}_p)E^p(\xi^\alpha), \quad E^p = \exp(ikk_\alpha^{(p)}\xi^\alpha) \end{aligned} \tag{7.2}$$

It is assumed everywhere here that $k = k_{\text{res}} = g_0/(2c^2) = 1/(2h_0\varkappa)$ which corresponds to the plane waves with the most rapidly increasing amplitude. The vectors $\mathbf{k}_p = \mathbf{e}_\alpha k^\alpha_{(p)}$, the sum of which is equal to zero, have the form

$$\mathbf{k}_1 = (1, 0, 0), \quad \mathbf{k}_2 = (-1/2, \sqrt{3}/2, 0), \quad \mathbf{k}_3 = (-1/2, -\sqrt{3}/2, 0)$$

It is clear that the term $\mathbf{r}_I$ is invariant under permutations of the vectors $\mathbf{k}_p$. The component $\mathbf{r}_{II}$ denotes the sum of higher, no longer resonant, harmonics, which increase more slowly. They appear as a result of the quadratic interaction of the plane waves. In the case of a single wave (Section 4),

$$A = B = A_0\operatorname{ch}\lambda t, \quad \lambda = g_0/(2c)$$

After substituting expressions (7.2) into Eq. (7.1) and reduction of similar terms accompanying the functions E^p, we obtain two equations for the complex functions A and B

$$\ddot{A} = g_0k\left(B - \frac{1}{2}A + \frac{3}{8}k\bar{B}^2\right), \quad \ddot{B} = g_0k\left(A - \frac{1}{2}B + \frac{3}{4}k\overline{AB}\right) \tag{7.3}$$

A complex conjugate is denoted by a bar.

The initial conditions have the form

$$A(0) = B(0) = A_0, \quad \dot{A}(0) = \dot{B}(0) = 0 \tag{7.4}$$

which corresponds to symmetric interaction of initially three plane waves of small amplitude A_0.

It is clear that, in particular, a real positive A_0 will be the best case of the organization of the acceleration of the shell. In this case, the solution of Eq. (7.3) will also be real and positive. In the case of a complex $A_0 = |A_0|e^{i\alpha}$, the effect of the argument α, generally speaking, leads to the absence of an intersection of the three lines of the maxima of the functions $\cos(kk_\beta^{(p)}\xi^\beta + \alpha)$. In the case of a real negative A_0, we obviously have a triple minimum at the point $\xi^\beta = 0$.

Eq. (7.3) have an energy integral

$$\frac{1}{2}(|\dot{A}|^2 + |\dot{B}|^2) - \frac{g_0k}{2}\left[B\bar{A} + A\bar{B} - \frac{1}{2}(|A|^2 + |B|^2) + \frac{3k}{8}(AB^2 + \bar{A}\bar{B}^2)\right] = E_0 \tag{7.5}$$

and can easily be solved numerically.

Suppose $\alpha = 0$. Then, taking account of the asymptotic forms when $t \to 0$ and $t \to \infty$, a simple approximation of the solution by elementary functions can also be proposed. In the dimensionless variables

$$a = kA, \quad b = kB, \quad \tau = \sqrt{g_0k}\,t$$

we have

$$a = a_0 + \frac{a_0\tau^2}{(2-(a_0/2)^{1/4}\tau)^2}, \quad b = a_0 + \frac{a_0\sqrt{2}\tau^2}{(2-(a_0/2)^{1/4}\tau)^2} - \frac{a_0(\sqrt{2}-1)\tau^2}{4}$$

The dimensionless time of departure to infinity is equal to $\tau_\infty = 2(2/a_0)^{1/4}$

Analysis of this solution shows that collapse occurs at points corresponding to the lines of the minimum of the functions $\cos(kk_\beta^{(p)}\xi^\beta)$ and material half-planes are formed which lag behind the main surface. These formations carry momentum, that acts on an obstacle. Comparison of the breakdown in the experiment described above with the data enables us to determine the quantity $l_{\text{res}} = 4\pi\kappa h_0$ and, consequently, to determine the constant of the material κ. As a result, we have $\kappa = 1.03$ in the case of copper.[7]

8. Conclusion

Besides the traditional model of an ideal incompressible fluids for an effective description of the phenomena associated with the explosive acceleration of metallic shells, the relatively simpler inertial and weakly elastic models can be used.

The inertial model only takes account of the inertial properties of an element of a shell which is accelerated by a pressure overfall and, when self-intersections arise, a model of an absolutely inelastic collision is used, which describes the formation of branching surfaces, lines or points. However, experiments which have been carried out on the accelerating of plates made of a sufficiently soft metal indicate that there is a systematic increase in the perturbations from a wavelength which is proportional to the initial thickness of the plate. This may be associated with a certain residual elasticity that manifests itself despite the large plastic deformations.

The proposed simple model of a weakly elastic shell refines the model of inertial accelerating by taking account of the longitudinal elasticity with a modulus of elasticity proportional to the magnitude of the load.

An asymptotic validation of the models of an inertial shell and a weakly elastic shell has been given on the basis of the dynamic behaviour of a three-dimensional thin layer of an incompressible liquid-crystal material. It has been shown that, in the case of a sufficiently smooth side of a shell, which is subjected to a load, the inertial model is obtained in the basic approximation. In the case of unevenness of the loaded side of the shell, which can be used in applications to produce, for example, favourable wave conditions, it is now necessary to use the model of a weakly elastic shell in the basic approximation.

We now touch the problem of explaining the physical features of the conversion of a solid plastic material into a liquid in thin layers under extremally high transverse pressure gradients. The investigations presented above, which have been compared with experiment, show that, under these conditions, the manifestation of liquid-crystal properties of the material is possible. A general theory of this kind of manifestations, based on a group classification of affine deformations, was proposed earlier in Ref. 15. The development of investigations in this direction would be of considerable interest.

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